# On the Uniform-Traffic Capacity of Single-Hop Interconnections Employing Shared Directional Multichannels

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***Abstract*—A shared directional multichannel (SDM) consists of a set of inputs and a set of outputs to which we connect transmitters and receivers, respectively. A signal placed at any given input reaches a subset of the outputs, and a channel is specified by the sets of outputs that are reachable from each input. A message is received successfully at an output of the channel if and only if it is addressed to the receiver connected to that output and no other signals reach that output at the same time. Constructive lower bounds as well as some upper bounds on the uniform-traffic capacity of SDM-based single-hop interconnections between a set of multitransmitter source stations and a set of multireceiver destination stations are derived. (Every source station is connected to every destination station through the channel.) A bidirectional interconnection among a set of stations would be obtained by representing each station as one source station and one destination station. Both randomized transmissions and deterministic scheduling are considered. It is shown that with randomized transmissions, SDM's that can be described as a collection of buses can perform as well as any other ones. With deterministic scheduling, however, the use of certain non-bus-oriented SDM's yields a much higher interconnection capacity.**

***Index Terms*—Shared directional multichannel, multiple access, fiberoptic interconnections, channel capacity, local area networks, concurrency.**

## I. Introduction

### A. Shared Communication Channels

Shared (multiple access) communication channels are used whenever one cannot afford to construct dedicated, point-to-point channels between every pair of user stations and does not wish to rely on other stations or dedicated switches for routing messages. Examples include the Ethernet local-area network [1], buses in computers, and radio networks.

Normally, a single shared channel is used to interconnect all stations, and the resulting network has the following characteristics:

- the required transmission rate on the channel must exceed both the desired data rate for a single transmission and the aggregate throughput of all station pairs, and
- with $N$ stations sharing a channel, the average (over stations) utilization of station hardware is at most $1/N$.

As the number of stations attached to a local area network and the network usage by each station increase, the required transmission rate is eventually dictated by the aggregate throughput of the network rather than by the peak data rate required for any single station. This forces users to pay for expensive hardware that is of no benefit to them, thereby making shared channels less attractive. It would, therefore, be nice to somehow decouple the required transmission rate from the aggregate throughput of the network. The question is how to do this while retaining the simplicity of single-hop connectivity through a passive medium.

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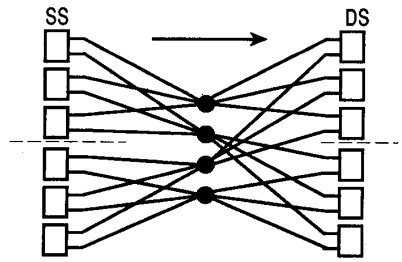


Fig. 1. Single-path bus-oriented interconnection network between source stations and destination stations. ($c_T = c_R = 2$; $N_S = N_D = 6$). There are $c_T \cdot c_R$ buses.

From the constraint of single-hop connectivity among all stations through a passive medium, it immediately follows that each station must be equipped with multiple transmitters or receivers if any decoupling is to be achieved. (Another option would be to use spread-spectrum techniques [2], but at least part of the circuitry would still have to operate at a rate exceeding the aggregate network throughput.) Throughout the correspondence, we will therefore explore passive, single-hop interconnections (SHI's) among stations with multiple transmitters and receivers. For generality and simplicity of presentation, we will present them as connecting a set of *source stations* (SS's) to a set of *destination stations* (DS's). Clearly, a bidirectional station can be represented as one SS and one DS. It should nevertheless be noted that the transmitters and receivers of each station are separate.

### B. Bus-Oriented Shared Multichannels

The simplest way of interconnecting user stations, each equipped with $c$ transmitters and receivers, is to construct $c$ shared channels, each interconnecting all stations through one of their transmitters and receivers [3], [4]. For a uniform traffic pattern, however, one can do better. (By "uniform traffic pattern" we mean an equal amount of traffic between every pair of stations.)

Let us set the capacity of a single shared channel to one unit, and let $c_T$ and $c_R$ denote the number of transmitters and receivers per station, respectively. In [5], [6], it was shown how to achieve a throughput of $c_T \cdot c_R$ for a uniform traffic pattern by constructing a collection of shared channels, each interconnecting a proper subset of the stations through one of their transmitters and receivers. Fig. 1 depicts such an interconnection. The idea is to split the source and destination stations into $c_R$ and $c_T$ groups of equal sizes, respectively, and to dedicate a unique bus to the connection of each group of source stations to each group of destination stations. Bus $(i, j)$ connects the $j$th transmitter of every source station in the $i$th group to the $i$th receiver of every destination station in the $j$th group. (The capacity is slightly lower whenever the groups cannot be of identical sizes due to integer constraints. The details, which are trivial, are omitted for brevity.)

The transmission rate required with this interconnection is only $1/(c_T \cdot c_R)$ of the aggregate network throughput. With $c_T = c_R = c$, the average utilization of station hardware grows to $c/N$.

We refer to these interconnections as "bus-oriented" because the sets of receivers that can hear any two transmitters are either identical or disjoint. Indeed, we were able to describe such an

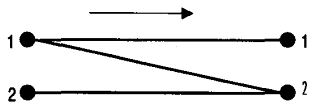


Fig. 2. A shared directional multichannel represented as a bipartite graph. Note that the subsets of receivers that can hear two transmitters are neither identical nor disjoint.

interconnection as a collection of conventional shared channels. Bus-oriented interconnections were also studied in [7] in the context of permutation networks. There, the number of buses is always equal to the number of stations. The uniform-traffic capacity of multihop bus-oriented interconnections was studied in [8], [9].

*Lemma 1 [5]:* The maximum (over topologies) uniform-traffic capacity of a bus-oriented single-hop interconnection among stations, each with $c_T$ transmitters and $c_R$ receivers, is $c_T \cdot c_R$.

This was proved [5] and [6] under certain symmetry assumptions, which were later relaxed in [8]. For a detailed study of bus-oriented single-hop interconnections, see [5], [6]. For a discussion of fiber-optic implementation of these interconnections, see [5], [10], [11].

### C. The Shared Directional Multichannel

A shared directional multichannel (SDM, for short) consists of a set of inputs and a set of outputs to which we connect transmitters and receivers, respectively. A signal placed at any given input reaches a subset of the outputs. A channel is specified by the subsets of outputs reached by signals at the different inputs.

Bus-oriented interconnections are SDM's. In general, however, an SDM does not adhere to the "bus-oriented" constraints, since the subsets of receivers that can hear some two transmitters need not be identical or disjoint. An SDM can be conveniently described as a bipartite graph $(U, V, E)$ with $U$ and $V$ representing inputs and outputs, respectively, and $(u \in U, v \in V) \in E$, if and only if a signal placed at input $u$ reaches output $v$. Fig. 2 depicts an SDM represented as a bipartite graph.

The channel adheres to the following rules.

1) A message transmitted into any given input of the channel reaches all outputs connected to it (and the receivers connected to those).
2) A message is received successfully by a receiver at an output of the channel, if and only if it is addressed to that receiver and no other signals reach that output at the same time.

The shared directional multichannel was introduced in [5], motivated by fiber-optic technology. (An SDM can be constructed easily using fibers and transmissive star couplers. The latter is a passive fiber-optic element with a number of inputs and a number of outputs, such that a signal presented at an input appears at all outputs but not at the other inputs [12].) The use of an SDM for connecting multitransmitter source stations to multireceiver destination stations was also suggested in [5], along with some specific designs and performance analysis.

In this correspondence, we study the throughput of SDM-based SHI's for a uniform traffic pattern. We present for the first time specific SHI's and transmission schedules which achieve a capacity that grows with the number of stations, and also derive some upper bounds on capacity.

In Section II, we discuss randomized transmissions. Section III focuses on deterministically-scheduled transmissions, and Section IV summarizes the correspondence.

## II. Maximum Throughput with Randomized Transmissions

Consider an SDM with $t$ inputs and $r$ outputs. Each input is connected to $d_T$ outputs and each output is connected to $d_R$ inputs. $(t \cdot d_T \equiv r \cdot d_R \cdot)$ We assume a slotted time system with single-slot messages, and the channel is operated as follows. In each time slot, each transmitter transmits with probability $p$. Whenever it transmits, the destination address is chosen at random and with equal probabilities from among the $d_T$ candidates. (This is a uniform traffic pattern of sorts.) The transmission process is independent from transmitter to transmitter and from slot to slot. As was stated earlier, a message is received successfully, if and only if it is transmitted and the intended recipient cannot hear any other transmissions in the same time slot.

*Lemma 2:* The maximum (over $p$) throughput of any such SDM is

$$\frac{1}{e} \cdot \frac{r}{d_T} \le S_{\max} \le 0.5 \cdot \frac{r}{d_T}, \qquad d_R \ge 2.$$

*Proof:*

$$\Pr\{i\text{th receiver receives a transmission in a given time slot}\} = d_R \cdot \frac{p}{d_T}(1-p)^{d_R - 1}.$$

Multiplying this by the total number of receivers, $r$, yields the aggregate throughput. The latter is maximized by setting $p = 1/d_R$, yielding

$$S_{\max} = \frac{r}{d_T} \cdot \left(1 - \frac{1}{d_R}\right)^{d_R - 1}.$$

Therefore,

$$\frac{1}{e} \cdot \frac{r}{d_T} \le S_{\max} \le 0.5 \cdot \frac{r}{d_T}, \qquad d_R \ge 2.$$

□

*Corollary 1:* With an unslotted system, $S_{\max} \ge (1/2e) \cdot (r/d_T)$. (In an unslotted system, transmissions may begin at any time, and transmission commencements constitute a Poisson process. We still assume message length to be fixed at one unit.)

Let us now use an SDM in constructing an SHI between a set of SS's and a set of DS's. We specify the channel such that for each (SS, DS) pair there are $k$ different (transmitter, receiver) pairs through which they can communicate. (For two (transmitter, receiver) pairs to be different, it suffices that either the transmitters or the receivers be different.) In the bipartite graph description, we assume that all input vertices have equal outdegrees $d_T$, and all output vertices have equal indegrees $d_R$. We refer to this as an *equal-degree, k-path SHI.* It is also assumed that all the transmitters of an SS can operate independently, as can the receivers of a DS.

*Lemma 3:* The maximum throughput of *any* $k$-path, equal-degree SHI connecting $N_s$ source stations, each with $c_T$ transmitters, to $N_D$ destination stations, each with $c_R$ receivers, for randomized transmissions and a uniform traffic pattern is

$$\frac{1}{e} \cdot \frac{c_T \cdot c_R}{k} \le S_{\max} \le 0.5 \cdot \frac{c_T \cdot c_R}{k} \cdot \qquad (N_S \ge c_R,\ N_D \ge c_T).$$

*Proof:* Follows directly from Lemma 2 with the following substitutions:

$$t = N_S \cdot c_T;\ r = N_D \cdot c_R;\ d_T = k \cdot N_D/c_T;\ d_R = k \cdot N_S/c_R.$$

□

Let us now restrict the discussion to the case of $k = 1$ (single-path SHI), and consider the situation wherein an SS can operate at most one of its transmitters in any given slot, and a DS can receive at most one transmission in any given slot. Each receiver is nevertheless assumed to be capable of independently deciding whether a transmission that it hears is receivable (no collision), and whether or not a receivable transmission is intended for its DS. Therefore, whenever the receivers of a DS hear at least one receivable transmission that is intended for their DS, one of those transmissions (chosen at random) is received. Each SS is assumed to transmit with probability $(p \cdot c_T)$ in each time slot; the transmitter is selected at random and the destination is selected at random from among those that can hear the selected transmitter. To calculate the throughput, observe the following.

1) A receiver can hear at most one transmitter of any given source station. Therefore, the reception process at a given receiver is not affected by a dependence between the transmission processes of different transmitters within the same SS.
2) The subsets of source *stations* that can reach two receivers of the same DS are disjoint. Consequently, the message arrival processes at two such receivers are independent. (Only true for $k = 1$.)

From 1, it follows that the probability that a given receiver hears a receivable transmission which is intended for its DS is

$$S_R = d_R \cdot \frac{p \cdot c_T}{N_D}\left(1 - \frac{p \cdot c_T}{c_T}\right)^{d_R - 1} = d_R \cdot \frac{p}{d_T}(1-p)^{d_R-1};$$

i.e., the same as in the previous case.

From 2), it follows that the throughput of a destination station is

$$S_{DS} = 1 - 1(1 - S_R)^{c_R},$$

and the aggregate throughput is thus,

$$S = N_D \cdot S_{DS} = N_D \cdot \left[1 - \left(1 - d_R \cdot \frac{p}{d_T}(1-p)^{d_R-1}\right)^{c_R}\right].$$

This is maximized by setting $p = 1/d_R$, yielding (for $d_R \gg 1$)

$$S_{\max} = N_D\left[1 - \left(1 - \frac{1}{e \cdot d_T}\right)^{c_R}\right].$$

Normally, $N \gg c$. Consequently, $e \cdot d_T \gg c_R$ and $S_{\max}$ is approximately $(1/e) \cdot c_T \cdot c_R$, which was the result for $k = 1$ with independently-operated transmitters and receivers. (In this situation, the probability of two or more receivers of the same station hearing receivable packets intended for them in the same time slot is negligible.)

In summary, we have seen that with randomized transmissions, a uniform traffic pattern and independent operation of distinct transmitters and receivers of any given station, all $k$-path equal-degree SHI's perform equally well. Specifically, the bus-oriented ones which are simplest to construct and operate are as good as the more general ones. Viewed differently, however, this allows the designer to incorporate other considerations into the design. For example, if the traffic pattern can be described as a sum of a uniform traffic pattern and a sparse nonuniform pattern, one could design the SHI to best accommodate the nonuniform component without altering the performance for the uniform one. For more details, see [5].

## III. Deterministically-Scheduled Transmissions

In this section, we show that it is possible to construct SDM-based SHI's with deterministic transmission schedules that greatly outperform the bus-oriented interconnections for a uniform traffic pattern. We again consider single-path SHI's.

Instead of computing throughput directly, we will assume that each SS has one message for every DS, and compute the number of time slots required for all the messages to be received successfully. This will be referred to as the *length* $l(X)$ of the transmission schedule $X$. The throughput is simply the total number of messages divided by $l(X)$.

For simplicity of notation, we make the following substitutions for variables used in previous sections: $c_T = a$, $c_R = b$; $N_S = m$, $N_D = n$. Other variables used here are unrelated to those in previous sections.

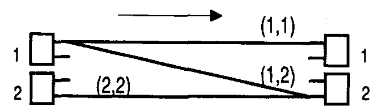


Fig. 3. Compatibility of wiring and schedule: since $W_1(1, 1) = W_1(1, 2)$ $(= 1)$ and $W_2(1, 2) = W_2(2, 2)(= 2)$, transmissions from $SS_1$ to $DS_1$ and from $SS_2$ to $DS_2$ must not be scheduled for the same slot. (Collision at $DS_2$.)

### A. Problem Statement and Preliminary Observations

Consider $m$ source stations $S = \{s_1, \cdots, s_m\}$ communicating with $n$ destination stations $D = \{d_1, \cdots, d_n\}$ in the following way: each station in $S$ has $a$ transmitters which we index by a set $T$, $|T| = a$, and each station in $D$ has $b$ receivers, indexed by $R$, $|R| = b$.

For each pair of stations $s \in S$ and $d \in D$, exactly one transmitter of $s$ is connected to exactly one receiver of $d$. Denote the index of that transmitter by $W_1(s, d) \in T$, and the index of that receiver by $W_2(s, d) \in R$. Let $W$ be the $m \times n$ matrix indexed by $S \times D$, whose entries are $W(s, d) = (W_1(s, d), W_2(s, d))$. $W$ is called the *Wiring matrix.*

Messages are transmitted at discrete time slots, and are all one slot in duration. The reception rules are those of the SDM.

We would like to devise transmission schedules in which each $s \in S$ will communicate successfully with each $d \in D$ at least once. Formally, a *transmission schedule* is an $m \times n$ matrix $X$ indexed by $S \times D$ whose entries $X(s, d)$ have time-slots as values.

A schedule $X$ is *compatible* with a wiring $W$, if and only if for all $(s, d) \in S \times D$, $s$ communicates successfully with $d$ at time $X(s, d)$. Formally, a schedule $X$ is compatible with a wiring $W$ iff for any two pairs $(s_1, d_1) \neq (s_2, d_2) \in S \times D$ the following holds: if both $W_1(s_1, d_1) = W_1(s_1, d_2)$ and $W_2(s_1, d_2) = W_2(s_2, d_2)$, then $X(s_1, d_1) \neq X(s_2, d_2)$. Fig. 3 illustrates the compatibility rules.

Let $f(a, b; m, n)$ denote the minimal schedule length for which there exist compatible wiring $W$ and transmission schedule $X$. By transposing both transmission and wiring matrices, it is clear that $f(a, b; m, n) = f(b, a; n, m)$.

We next give some upper and lower bounds on $f(a, b; m, n)$. These translate to lower and upper bounds on capacity, respectively.

### B. Upper Bounds on $f(a, b; m, n)$

We derive our bounds by combining an explicit construction (Lemma 4) with a recursive argument (Lemma 5).

*Lemma 4:* For any integers $1 \leq r \leq k$,

$$f\left(r, 1; \binom{k}{r-1}, r^k\right) \leq r^k.$$

*Proof:* Let $\mathbf{Z}_r = \{0, 1, \cdots, r-1\}$ denote the residues modulo $r$, and $\mathbf{Z}_r^k = \{(x_1, \cdots, x_k) : x_i \in \mathbf{Z}_r\}$. The inner product of two vectors $x = (x_1, \cdots, x_k)$, $y = (y_1, \cdots, y_k) \in \mathbf{Z}_r^k$ is given by $x \cdot y = \sum_{i=1}^k x_i y_i$ where all operations are carried out in $\mathbf{Z}_r$.

Now let $S$ be the set of 0–1 vectors in $\mathbf{Z}_r^k$ with exactly $(r-1)$ "1's", and $D = \mathbf{Z}_r^k$. The wiring matrix is defined as follows: for each $(s \in S, d \in D)$, let $W_1(s, d) = s \cdot d \in \mathbf{Z}_r$. (Operations in $\mathbf{Z}_r$.) $W_2(s, d)$ is, of course, identically 1. The scheduling matrix is defined by $X(s, d) = s + d \in \mathbf{Z}_r^k$. (Operations in $\mathbf{Z}_r^k$.) It remains to be shown that $X$ and $W$ are compatible: Suppose $(s_1, d_1), (s_2, d_2) \in S \times D$. We have to show that if both $W_1(s_1, d_1) = W_1(s_1, d_2)$, and $X(s_1, d_1) = X(s_2, d_2)$, then $(s_1, d_1) = (s_2, d_2)$. But the first two equalities imply that $s_1 \cdot d_1 = s_1 \cdot d_2$ and $s_1 + d_1 = s_2 + d_2$. Hence,

$$\begin{aligned} 0 &= s_1 \cdot (d_2 - d_1) = s_1 \cdot (s_1 - s_2) \\ &= r - 1 - s_1 \cdot s_2 = r - 1 - \left| s_1 \bigcap s_2 \right|, \end{aligned}$$

where $s_1 \bigcap s_2$ is the number of common 1's in $s_1$ and $s_2$. Therefore, $\left| s_1 \bigcap s_2 \right| \equiv r - 1 \pmod r$ and so $s_1 = s_2$ and $d_1 = d_2$. □

Fig. 4 depicts the wiring (only $W_1$, since $b = 1$) and transmission schedule matrices for a (2, 1; 4, 16) SHI. (The matrices are transposed for formatting convenience.) Time slot number 5 is highlighted in the figure. "+" is used to denote the connections used for the actual transmissions, and "−" marks the stray destinations, i.e., those that are not addressees yet hear a transmission. Observe that the destinations of the actual transmissions hear no other transmissions. In general, this must only hold for the individual receivers that receive the desired transmission, and need not be true of the DS as a whole. In this example, however, a DS has a single receiver.

*Corollary 2:*

$$f(r, 1;\ n, n) \le C(r) \cdot \frac{n^2}{(\log_2 n)^{r-1}},$$

where $C(r) \le r^2((r-1)\log_2 r)^{r-1}$.

*Proof:* By appropriately duplicating the wiring and schedule matrices it is clear (when $n \ge m$) that $f(r, 1;\ n, n) \le \lceil \frac{n}{m} \rceil \cdot f(r, 1;\ m, n)$. Combining this observation and Lemma 4, with $k = \lceil \log n / \log r \rceil$ and $m = \binom{k}{r-1}$, we obtain

$$\begin{aligned} f(r, 1;\ n, n) &\le \left\lceil \frac{n}{\binom{k}{r-1}} \right\rceil \cdot f\left(r, 1;\ \binom{k}{r-1}, n\right) \\ &\le \left\lceil \frac{r^k}{\binom{k}{r-1}} \right\rceil f\left(r, 1;\ \binom{k}{r-1}, r^k\right) \\ &\le \left\lceil \frac{r^k}{\binom{k}{r-1}} \right\rceil \cdot r^k \\ &\le r^2((r-1)\log_2 r)^{r-1} \frac{n^2}{(\log_2 n)^{r-1}}. \end{aligned}$$ □

Given *any* two SHI's, each with compatible wiring and schedule (and thus known values of $f$), the following lemma will enable us to produce a constructive upper bound on $f$ for an SHI with larger parameter values.

*Lemma 5:* Let $a = a_1 \cdot a_2$, $b = b_1 \cdot b_2$, $m = m_1 \cdot m_2$, and $n = n_1 \cdot n_2$. Then,

$$f(a, b;\ m, n) \le f(a_1, b_1;\ m_1, n_1) \cdot f(a_2, b_2;\ m_2, n_2).$$

*Proof:* For $i \in \{1, 2\}$ let $S_i$, $D_i$, $T_i$, $R_i$ satisfy $|S_i| = m_i$, $|D_i| = n_i$, $|T_i| = a_i$, $|R_i| = b_i$, where, as before, $T_i$ indexes the transmitters of each station in $S_i$ and $R_i$ indexes the receivers of each station in $D_i$. Let $W^i = (W_1^i, W_2^i)$ and $X^i$ be compatible wiring and scheduling matrices for the communication between $S_i$ and $D_i$, such that $W_1^i(s_i, d_i) \in T_i$ and $W_2^i(s_i, d_i) \in R_i$ for $(s_i, d_i) \in S_i \times D_i$. Assume that $W^i$, $X^i$ is an optimal-length scheme, i.e., $l(X^i) = f(a_i, b_i;\ m_i, n_i)$.

Let $S = S_1 \times S_2$, $D = D_1 \times D_2$, $T = T_1 \times T_2$, and $R = R_1 \times R_2$. We shall construct wiring and scheduling matrices for communication between $S$ and $D$, where $T$ indexes the transmitters of each station in $S$, and $R$ indexes the receivers of each station in $D$. For a pair

| d \ s | 1 0 0 0 | 0 1 0 0 | 0 0 1 0 | 0 0 0 1 |
|---|---|---|---|---|
| 0 0 0 0 | 0 | -0- | 0 | -0- |
| 0 0 0 1 | 0 | +0+ | 0 | 1 |
| 0 0 1 0 | 0 | -0- | -1- | -0- |
| 0 0 1 1 | 0 | -0- | -1- | 1 |
| 0 1 0 0 | 0 | 1 | 0 | +0+ |
| 0 1 0 1 | 0 | 1 | 0 | 1 |
| 0 1 1 0 | 0 | 1 | -1- | -0- |
| 0 1 1 1 | 0 | 1 | +1+ | 1 |
| 1 0 0 0 | -1- | -0- | 0 | -0- |
| 1 0 0 1 | -1- | -0- | 0 | 1 |
| 1 0 1 0 | -1- | -0- | -1- | -0- |
| 1 0 1 1 | -1- | -0- | 1 | 1 |
| 1 1 0 0 | -1- | 1 | 0 | -0- |
| 1 1 0 1 | +1+ | 1 | 0 | 1 |
| 1 1 1 0 | -1- | 1 | -1- | -0- |
| 1 1 1 1 | -1- | 1 | -1- | 1 |

(a)

| d \ s | 1 0 0 0 | 0 1 0 0 | 0 0 1 0 | 0 0 0 1 |
|---|---|---|---|---|
| 0 0 0 0 | 1 0 0 0 | 0 1 0 0 | 0 0 1 0 | 0 0 0 1 |
| 0 0 0 1 | 1 0 0 1 | (0 1 0 1) | 0 0 1 1 | 0 0 0 0 |
| 0 0 1 0 | 1 0 1 0 | 0 1 1 0 | 0 0 0 0 | 0 0 1 1 |
| 0 0 1 1 | 1 0 1 1 | 0 1 1 1 | 0 0 0 1 | 0 0 1 0 |
| 0 1 0 0 | 1 1 0 0 | 0 0 0 0 | 0 1 1 0 | (0 1 0 1) |
| 0 1 0 1 | 1 1 0 1 | 0 0 0 1 | 0 1 1 1 | 0 1 0 0 |
| 0 1 1 0 | 1 1 1 0 | 0 0 1 0 | 0 1 0 0 | 0 1 1 1 |
| 0 1 1 1 | 1 1 1 1 | 0 0 1 1 | (0 1 0 1) | 0 1 1 0 |
| 1 0 0 0 | 0 0 0 0 | 1 1 0 0 | 1 0 1 0 | 1 0 0 1 |
| 1 0 0 1 | 0 0 0 1 | 1 1 0 1 | 1 0 1 1 | 1 0 0 0 |
| 1 0 1 0 | 0 0 1 0 | 1 1 1 0 | 1 0 0 0 | 1 0 1 1 |
| 1 0 1 1 | 0 0 1 1 | 1 1 1 1 | 1 0 0 1 | 1 0 1 0 |
| 1 1 0 0 | 0 1 0 0 | 1 0 0 0 | 1 1 1 0 | 1 1 0 1 |
| 1 1 0 1 | (0 1 0 1) | 1 0 0 1 | 1 1 1 1 | 1 1 0 0 |
| 1 1 1 0 | 0 1 1 0 | 1 0 1 0 | 1 1 0 0 | 1 1 1 1 |
| 1 1 1 1 | 0 1 1 1 | 1 0 1 1 | 1 1 0 1 | 1 1 1 0 |

(b)

Fig. 4. Wiring (a) and schedule (b) matrices (transposed) for an SDM-based SHI with $a = 2$, $b = 1$; $m = 4$, $n = 2^4 = 16$. Only $W_1$ is shown. "+" and "-" highlight desired and stray transmissions in time-slot 5, respectively.

of stations $s = (s_1, s_2) \in S_1 \times S_2$, $d = (d_1, d_2) \in D_1 \times D_2$, we define

$$\begin{aligned} W(s, d) &= (W_1(s, d), W_2(s, d)) \\ &= ((W_1^1(s_1, d_1), W_1^2(s_2, d_2)), \\ &\quad (W_2^1(s_1, d_1), W_2^2(s_2, d_2))) \end{aligned}$$

and $X(s, d) = \left(X^1(s_1, d_1), X^2(s_2, d_2)\right)$.

We claim that $W$ and $X$ are compatible: Suppose $\overline{s} = (\overline{s_1}, \overline{s_2}) \in S$, $\overline{d} = (\overline{d_1}, \overline{d_2}) \in D$ is a pair of stations such that $X(\overline{s}, \overline{d}) = X(s, d)$, $W_1(s, d) = W_1(s, \overline{d})$, and $W_2(\overline{s}, \overline{d}) = W_2(s, \overline{d})$. We have to show that $s = \overline{s}$ and $d = \overline{d}$. Indeed, our assumptions imply that $X^i(\overline{s_i}, \overline{d_i}) = X^i(s_i, d_i)$, $W_1^i(s_i, d_i) = W_1^i(s_i, \overline{d_i})$, and $W_2^i(\overline{s_i}, \overline{d_i}) = W_2^i(s_i, \overline{d_i})$ for $i \in \{1, 2\}$. Since $W^i$, $X^i$ are compatible, this implies that $s_i = \overline{s_i}$, $d_i = \overline{d_i}$ for $i \in \{1, 2\}$, hence, $(s, d) = (\overline{s}, \overline{d})$. Therefore, $W$ and $X$ are compatible and so

$$\begin{aligned} f(a, b;\ m, n) &\le l(X) = l(X_1) \cdot l(X_2) \\ &= f(a_1, b_1;\ m_1, n_1) \cdot f(a_2, b_2;\ m_2, n_2). \end{aligned}$$ □

Combining Corollary 2 and Lemma 5, we obtain the following.

*Corollary 3:*

$$f(a, b;\ n, n) \le C_1(a, b) \cdot \frac{n^2}{(\log_2 n)^{a+b-2}},$$

where $C_1(a, b) = 2^{a+b+2} \cdot C(a) \cdot C(b)$.

*Proof:*

$$f(a, b;\ n, n) \le f\left(a, 1;\ \lceil\sqrt{n}\rceil, \lceil\sqrt{n}\rceil\right) \cdot f\left(1, b;\ \lceil\sqrt{n}\rceil, \lceil\sqrt{n}\rceil\right) \le 2^{a+b+2} \cdot C(a) \cdot C(b) \cdot \frac{n^2}{(\log_2 n)^{a+b-2}}. \qquad \square$$

*Remark 1:* Corollaries 5 and 7 are, of course, interesting mainly when $n$ is large. When $a$, $b$ are large the following construction may also be useful: Assemble an $(a, b;\ m, n)$ SHI as a collection of $k$ $(a/k, b/k;\ m, n)$ SHI's, each of which is constructed as implied by the foregoing derivations. These can be operated concurrently, with the $i$th one executing all the $i \pmod k$ slots of the schedule. Therefore, it is always true that

$$f(a, b;\ m, n) \le \frac{1}{k} \cdot f(\lceil a/k\rceil, \lceil b/k\rceil;\ m, n).$$

*Remark 2:* The bound in Corollary 3 is valid for all $n$, but the factor $C_1(a, b)$ can usually be significantly improved by using Lemmas 4 and 5 directly. For example,

$$f\left(2, 2; k2^k, k2^k\right) \le f\left(2, 1;\ k, 2^k\right) \cdot f\left(1, 2;\ 2^k, k\right) \le 2^{2k}.$$

Thus, taking $n = k2^k$, we obtain

$$f(2, 2;\ n, n) \le \frac{n^2}{(\log_2 n - \log_2 \log_2 n)^2}.$$

To illustrate this example, consider a network with 160 stations, each equipped with two transmitters and two receivers. For a uniform traffic pattern, this network can carry 25 concurrent transmissions. Thus, 10 Mb/s transmission rates would produce a 250 Mb/s network! (Similarly, 16 concurrent transmissions for 64 stations.)

*Remark 3:* Let $g(r, k)$ denote the maximal cardinality of a set $A \subset \mathbb{Z}_r^k$ which satisfies $x \cdot (x - y) \ne 0$ for any distinct $x, y \in A$. In the proof of Lemma 4, we showed by construction that $g(r, k) \ge \binom{k}{r-1}$. Since this proof relied only on the fact that for any two distinct SS's, say $s_1$ and $s_2$, $s_1 \cdot (s_1 - s_2) \ne 0$, it also follows that $f\left(r, 1;\ g(r, k), r^k\right) \le r^k$. Since the uniform-traffic capacity of such an $\left(r, 1;\ g(r, k), r^k\right)$ SHI and of the $\left(r, 1;, r^k, r^k\right)$ SHI derived from it is $g(r, k)$, it would be interesting to know more about $g(r, k)$. We think that $g(r, k) = O\left(k^{r-1}\right)$ but can prove it only when $r$ is an integer power of a prime.

*Proposition:* If $q = p^l$ for prime $p$, then $g(q, k) = O\left(k^{q-1}\right)$ as $k \to \infty$.

*Proof:* Suppose $A = \{a_1, \cdots, a_s\} \subset \mathbb{Z}_r^k$ satisfies $a_i \cdot (a_i - a_j) \ne 0$ for all $i \ne j$. Let $\mathbf{Q}[x_1, \cdots, x_k]$ denote the ring of polynomials with rational coefficients in the variables $x_1, \cdots, x_k$. For $1 \le i \le s$ define the polynomials $u_i(x_1, \cdots, x_k) \in \mathbf{Q}[x_1, \cdots, x_k]$ by $u_i(x) = \binom{a_i \cdot (a_i - x) - 1}{q-1}$. We shall need the following lemma.

*Lemma [13]:* Let $b$ be an integer, then $\binom{b-1}{q-1} \equiv 0 \pmod p$ iff $b \not\equiv 0 \pmod q$. $\square$

We now show that $u_1, \cdots, u_s$ are linearly independent in $\mathbf{Q}[x_1, \cdots, x_k]$. Suppose to the contrary that $\sum_{i=1}^s \lambda_i u_i = 0$ and not all $\lambda_i$'s are zeros. Clearly we may assume that all $\lambda_i \in \mathbf{Z}$ and $\gcd(\lambda_1, \cdots, \lambda_s) = 1$. Now the Lemma implies that $u_i(a_j) \equiv 0 \pmod p$ iff $i \ne j$, hence for any $1 \le j \le s$, $0 = \sum_{i=1}^s \lambda_i u_i(a_j) \equiv \lambda_j u_j(a_j) \pmod p$, and so $\lambda_j \equiv 0 \pmod p$ contradicting $\gcd(\lambda_1, \cdots, \lambda s) = 1$. Thus $u_1, \cdots, u_s$ are linearly independent polynomials of degree $\le q - 1$ and so

$$s \le \dim\{w \in \mathbf{Q}[x_1, \cdots, x_k] : \deg w \le q - 1\} = \sum_{i=0}^{q-1} \binom{k + i - 1}{i} = O\left(k^{q-1}\right). \qquad \square$$

The proof uses the spaces of polynomials technique (see [14]). It seems that the asymptotic behavior of $g(r, k)$ (for $r$ not a power of a prime) is unknown even when $A$ is assumed to contain only $0-1$ vectors (see [13] and [14]).

### C. Lower Bounds on $f(a, b;\ m, n)$

For the general case, we are unable to establish tight lower bounds. However, we can offer some insight.

*Lemma 7:* $f\left(2, 1;\ k, 2^k\right) = 2^k$.

*Proof:* From Lemma 4, it follows that $f\left(2, 1;\ k, 2^k\right) \le 2^k$. If some SS transmits in a slot with both its transmitters, no other transmissions can be received in that slot. Therefore, at most $k$ concurrent transmission are possible, which yields $f\left(2, 1;\ k, 2^k\right) \ge 2^k$. $\square$

*Lemma 8:* Let $a = 2$, $b = 1$, and $n = 2^k$. If there is a compatible $(W, X)$ pair in which all source stations transmit successfully in every slot then $m \le k + 1$.

*Proof:* Let us select a row in $W$ that has at least half of its $W_1$ entries equal to one. (At least $2^{k-1}$ such entries.) Clearly, the corresponding entries in $X$ must all have different values (time slots). Moreover, none of these values may appear in $X$ in any of the columns in which the ones (in $W$) were found. Thus, for any $t$, $f\left(2, 1;\ t, 2^k\right) \ge 2^{k-1} + f\left(2, 1;\ t - 1, 2^{k-1}\right)$. Applying this inequality repeatedly $k$ times and noting that $f(2, 1;\ 2, 1) = 2$ we get $f\left(2, 1;\ k + 2, 2^k\right) \ge 1 + 2^k > 2^k$. But this means that not all SS's are transmitting in every slot, since there are only $(k + 2)2^k$ messages. $\square$

In [9], the $\left(2, 1;\ k, 2^k\right)$ interconnection presented here and in [15] was augmented with another SS to form a $\left(2, 1;\ k + 1, 2^k\right)$ interconnection with capacity $k + 1$, thus matching the bound of Lemma 8. (The wiring of the additional SS to any given DS is based on the parity of the binary string representing the DS number.)

## IV. Summary

Collections of bus-oriented shared channels, each connecting a subset of multitransmitter source stations to a subset of multireceiver destination stations, have been shown to provide uniform-traffic capacity that increases quadratically with the number of transmitters and receivers per station. With $c$ transmitters and receivers per station, the transmission rate is therefore only $1/c^2$ of the aggregate network throughput. The separation between such channels can be spatial, spectral, etc. or any combination thereof. Unfortunately, however, the capacity does not increase with an increase in the number of stations.

Transmission media in which directional couplers can be easily implemented lend themselves to the construction of shared directional multichannels, which can provide arbitrary passive single-hop interconnections between a set of multitransmitter source stations and a set of (possibly the same) multireceiver destination stations. (This includes the bus-oriented interconnections as a special case.)

In this correspondence, we explored the capacity of such interconnections for a uniform traffic pattern. We showed that with randomized transmissions, the extra flexibility offers no direct advantage over the bus-oriented interconnections. However, with a deterministic transmission schedule and $N$ stations, it is possible to achieve a capacity of at least $\left(1/((c-1)\log_2 c)^{2c-2}\right) \cdot (\log_2 N)^{2c-2}$ concurrent noninterfering transmissions. ($c$ is the number of transmitters and receivers per station.) Thus, for any fixed value of $c$, the capacity increases with an increase in the number of stations! For fixed $N$, the previous function has a maximum for some value of $c$. However, the actual capacity increases at least linearly with $c$, since we can always construct an SDM-based SHI for a large value of $c$ in the form of several smaller SHI's, each utilizing a subset of transmitters and receivers of every station.

To give the results a practical flavor, consider a network with 160 stations, each equipped with two transmitters and two receivers. For a uniform traffic pattern, this network can carry 25 concurrent transmissions. Thus, 10 Mb/s transmission rates would produce a 250 Mb/s network! (Similarly, 16 concurrent transmissions for 64 stations.)

We have thus demonstrated that shared directional multichannels can offer important advantages over bus-oriented interconnections. As such, they are worthy of further study. Topics might include access schemes, design and performance for nonuniform traffic patterns, and wiring. Some of these issues were partly addressed in [5], and the possibility of efficient layout has been demonstrated in [16].

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# Growing Binary Trees in a Random Environment

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***Abstract*—A class of binary trees that grow in a random environment, where the state of the environment can change at every vertex of the trees is studied. The trees considered are single-type and two-type binary trees that grow in a two-state Markovian environment. For each kind of tree, the conditions on the environment process for extinction of the tree are determined, and the problem of calculating the expected number of vertices of the tree is addressed. Different ways of growing the trees are compared.**

***Index Terms*— Random trees, growing trees, random environment, splitting algorithms.**

## I. Introduction

Consider a growing tree of which each vertex generates additional vertices according to some probabilistic reproduction law. Growing trees arise naturally in many applications, such as searching and sorting [8], multiaccess comunication [2], and growth of populations [3], [4]. Often, the tree that arises is growing in presence of a stochastic process, the *random environment,* which determines the reproduction law of each vertex. In addition, the tree may consist of vertices of different types, and the reproduction law of each vertex may depend on the type of the vertex.

We study a class of binary trees that grow in a random environment, which arise in multiaccess communication when the communication channel is noisy [6], [9], [12]. In this case, the growing tree describes a splitting algorithm and the random environment corresponds to the noise process. The importance of the trees considered lies in the fact that they determine the stability of the algorithms.

Most previous studies of randomly growing trees do not assume the existence of a random environment, and are based on the assumption that the vertices reproduce independently of each other. Growing trees in a random environment were considered so far only in the context of branching processes in a random environment [4], with the restriction that the state of the environment can change only at every *generation,* so that vertices that belong to the same generation (and are of the same type) have always the same reproduction law [1].

The binary trees considered here are growing in a random environment where the state of the environment can change at every *vertex.* Thus, the reproduction law is chosen separately for each vertex of the tree, and vertices that belong to the same generation need not have

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